Laplace Transforms, Part 4:
Frequency and Phase Analysis

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Credit: 3 PDH

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Laplace Transforms in Design and Analysis of Circuits©
Part 4
by Tom Bertenshaw

Frequency and Phase Analysis

Domain of "s"

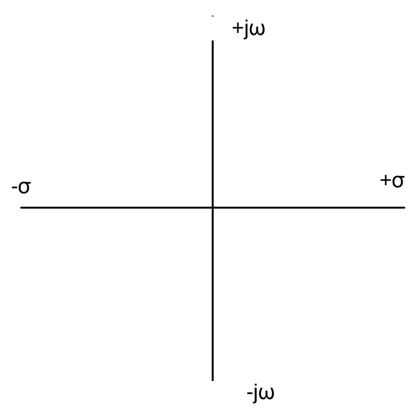
Suppose there is a quadratic pole in a transfer function, such as:

$$s^2 + 6s + 13$$

The roots of that expression are:

$$s = -3 \pm j2$$

Clearly for this quadratic the roots are complex, and we know from previous modules that the real part determines the time constant and the imaginary part represents the frequency of oscillation in rads. We eventually will plot these roots using a two dimensional plane wherein the real axis is defined as the horizontal axis and the imaginary axis is defined as vertical, each intersecting the other at the origin. However, for the present it is important to recognize that complex roots may be present as a reactive circuit's response to an impulse. At some point, we will need to plot the trajectory of the poles as a function of gain and frequency. With that in mind, the following is defined.



For the present, we will leave this topic, but we will return to it in later modules and use this coordinate system in pursuit of stability in circuit design. For reasons of stability, we will restrict the signs in the quadratic to be positive, i.e. the real parts of the roots lie to the left of the $j\omega$ axis. That is a general, and prudent, design constraint. A little thought about the behavior of $f(t)$ as a function of $e^{+\sigma t}$ provides the reason.

No matter how many iterations of finding the roots of a large quantity of quadratic equations where the roots are complex, we will consistently come to the conclusion that the domain of s is the entire complex plane. But specifically for our present purposes $s = -\sigma \pm j\omega$ is the general solution of a complex quadratic with roots that are to the left of the $j\omega$ axis.

The transient portion of the denominator of the transfer function generally is of the form:

$$(s+\sigma)^2 + \omega_d^2$$

And for the steady state response to an AC driver is:

$$s^2 + \omega_o^2$$

σ is a real number that is a function of the time constant, and ω is the frequency of oscillation in rads/s (when $\sigma \neq 0$ (in the example it is 3), ω is the damped frequency, (in the above case 2); a frequency less than its resonance of $\sqrt{13}$). The values of both $\sigma \& \omega$ arise from the values of the circuit components.

Extending this argument, consider the core variable in a general Laplace transform ($s = -\sigma \pm j\omega$):

$$e^{-st} = e^{-(\sigma \pm j\omega)t} = e^{-\sigma t} e^{\pm j\omega t}$$

Since exponents are unitless the units on both $\sigma \& \omega$ must be t^{-1}, and indeed, they are. Question: can you show this is true? Ponder: Do you see a shadow of relationship between a LaPlace transform and a Fourier transform from the expression $e^{-\sigma t} e^{\pm j\omega t}$?

The point of all this is that since the Laplace transform can be written as a function of $e^{-\sigma t} e^{\pm j\omega t}$, we can legitimately develop a method for expressing the response of the transfer function as a function of frequency, and by extension, its initial phase at a given frequency.

So far, we have examined the case where the roots are complex. For the case of real roots, with no imaginary component, those expressions will be first order roots of the form:

$$s+\sigma$$

Repeated real roots are an exception and they are of an order:

$$(s+\sigma)^m \quad m=2,3,\ldots\ldots.n$$

Repeated roots **do not** present the same degree of difficulty in frequency analysis as their inversion does when finding partial fraction expansions (PFE) for time domain analysis. Some excellent engineers that I have been associated with over the years will argue that exact repeated roots are not possible, so in the practical case they never have to be dealt with. The reasoning behind that is that no two time constants can ever contain components whose values are exactly the same to n decimal places, i.e., there will always be a slight amount of ambiguity in value, and the analyst can take advantage of that to use only non-repeated roots, regardless of how close any n roots are to each other. However, FAPP (For All Practical Purposes[1]) using n roots that are .1% apart make **no discernable** difference in the output from n repeated identical roots.

How to treat repeated roots when inverting into the time domain is another of those cases where you have to be aware that the method you choose may lead to ridiculous amplitudes. For the sake of prudence when we invert in this series of modules we will stick to identical roots and use the formal method of differentiation for PFE. Choosing methods is not a consideration when dealing with frequency analysis as there is no need for PFE.

Since any polynomial of degree 2 or greater can be factored into some combination of first order and/or second order terms the analysis techniques covered thus far in these modules are sufficient for our present purposes. In general, a transfer function will always be a combination of first order, second order and any repeated root factors in both the numerator and the denominator, and the second order roots will always be complex.

[1] John S. Bell, 1928-1990, Physicist extraordinaire

$$\frac{(s+z_1)(s+z_2)........(s+z_m)(s+z_r)^p\left((s+\sigma_1)^2+\omega_1^2\right)......\left((s+\sigma_m)^2+\omega_m^2\right)}{(s+p_1)(s+p_2)........(s+p_n)(s+p_r)^k\left((s+\alpha_1)^2+\omega_{d1}^2\right).....\left((s+\alpha_n)^2+\omega_{dn}^2\right)} \quad \leftarrow \text{Eq. 1}$$

Where $s = -z_x$ are the zeros of the function (causes the function to be zero), and $s = -p_x$ are the poles (like a telephone pole, causes the function to spike). It should be clear that if any roots are complex, a quadratic will appear, and all others will be first order roots or repeated roots that are real (repeated complex roots are also possible, but we will leave that case to the future).

Obviously, a frequency is a constituent of the complex case, but what of the other cases? It can be shown that the units of a time constant (RC or L/R) are time. It follows that the reciprocal of the time constant is frequency. The frequency associated with the reciprocal of the time constant is called the "break frequency", for reasons that will be apparent shortly.

The Transfer Function as a Logarithm

For the purposes of analysis or design, if we consider the behavior of a transfer function as a function of ω (or alternately, the reciprocal of RC or L/R) we will find that an accurate picture of amplitude versus frequency and phase versus frequency emerges.

First a digression. Recall that a decibel is defined as (P_{xx} is power in watts):

$$10\log\frac{P_{out}}{P_{in}}$$

Which can be re-written as:

$$10\log\frac{P_{out}}{P_{in}} = 10\log\frac{V_{out}^2}{V_{in}^2} = 20\log\frac{V_{out}}{V_{in}}$$

The above definition assumes that the input and output resistances are approximately equal, allowing the substitution of $\frac{V_{xx}^2}{R_{xx}}$ for P_{xx}. That is not an overly confining assumption in a passive circuit as it allows for maximum power transfer, a generally desirable design feature.

Any transfer function in the LaPlace domain is the ratio of:

$$\frac{N(s)}{D(s)}$$

(See Eq. 1) and that ratio can certainly be constructed to express the relationship of V_{out} to V_{in}. We can re-write the transfer function as:

$$20\log\frac{V_{out}}{V_{in}} = 20\log N(s) - 20\log D(s)$$

The Details – Creating Bode Plots

Very often it is convenient to be able to sketch the response of a circuit's output magnitude as a function of frequency. All reactive circuits possess the ability to vary the output as frequency changes. Recall that $X_c = \frac{-j}{\omega C}$ & $X_L = j\omega L,$ meaning both reactance's change value with frequency. So, if the circuit is reactive, to characterize the output you must specify a frequency or range of frequencies. Quite often we start at zero and proceed out towards infinity only as far as a practical measureable output is detectable. In practice, a sketch is usually sufficient to yield the desired information, however an accurate plot based on computation is first desirable in a learning environment to acquaint you with the consistent and recurring inaccurate envelopes inherent in the sketch technique. Generally the sketch is made on semi-log or log-log paper.

Assume a simple transfer function of the form:

$$\frac{K}{s+\sigma}$$

Where both K and σ are real scalers, re-write that to obtain the form:

$$\frac{K/\sigma}{\left(1+\frac{s}{\sigma}\right)}$$

Since the units on $\sigma = \frac{1}{t}$, let σ equal ω_b. Then we substitute $j\omega$ for s and the transfer function now looks like:

$$\frac{\frac{K}{\omega_b}}{\left(1+\frac{j\omega}{\omega_b}\right)}$$

From the Module 3, we found that $\sigma = \omega_b = \frac{1}{RC}$ (from an RC ckt) or $\frac{L}{R}$, (from an RCL ckt.) so as a designer you always retain control over that/those value(s).

$\left(1+\frac{j\omega}{\omega_b}\right)$ is complex, so we will convert that to polar notation – remember we are finding magnitude, so mag $\left(1+\frac{j\omega}{\omega_b}\right) \rightarrow \sqrt{1+\left|\frac{j\omega}{\omega_b}\right|^2} \rightarrow \sqrt{1+\frac{\omega^2}{\omega_b^2}}$, and finally:

$$\left(1+\frac{j\omega}{\omega_b}\right) = \sqrt{1+\frac{\omega^2}{\omega_b^2}}\ \angle \tan^{-1}\left(\frac{\omega}{\omega_b}\right)$$

Taking the logarithm of both sides, the magnitude of the transfer function from above is:

$$20\log\frac{V_{out}}{V_{in}} = 20\log K - 20\log\omega_b - 20\log\sqrt{1+\frac{\omega^2}{\omega_b^2}}$$

The phase angle as a function of frequency is $-\arctan\left(\frac{\omega}{\omega_b}\right)$ (why minus arctan?) Because the expression is in the denominator and the phases of the constants are both zero – as illustrated in the following example.) Phase angle is always taken to mean the phase of the output with respect to the input.

Lo-Pass Filter

As an example, consider:

$$\frac{10}{s+10}$$

For purposes of illustration the left hand side of transfer functions is omitted but understood to be $\frac{V_{out}}{V_{in}}$ for our present purposes. Later on, the left hand parameters may change and if/when it does the change will be identified.

Re-arranging, and letting $s = j\omega$:

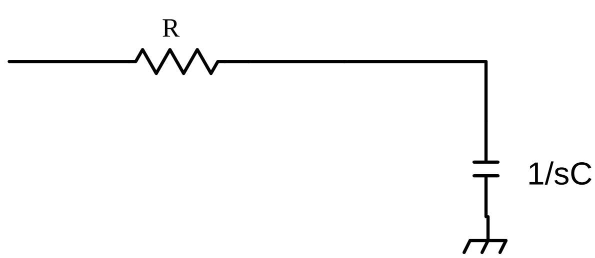
$$\frac{1}{1+\frac{s}{10}} \rightarrow \frac{1}{1+\frac{j\omega}{10}} \rightarrow \frac{1}{\sqrt{1+\frac{\omega^2}{100}}\angle \tan^{-1}\left(\frac{\omega}{10}\right)}$$

Converting to decibels:

$$Magnitude(\omega) = \frac{1}{\sqrt{1+\frac{\omega^2}{100}}} \rightarrow 20\log(1) - 20\log\left(\sqrt{1+\frac{\omega^2}{100}}\right)$$

$$phase(\omega) = \arctan(0) - \arctan\left(\frac{\omega}{10}\right)$$

It may be helpful to connect the transfer function to a circuit to help visualize what is going on here.



In the above circuit (a single pole lo-pass) assume V_{out} is taken across the capacitor and that $RC = .1$, in that case then:

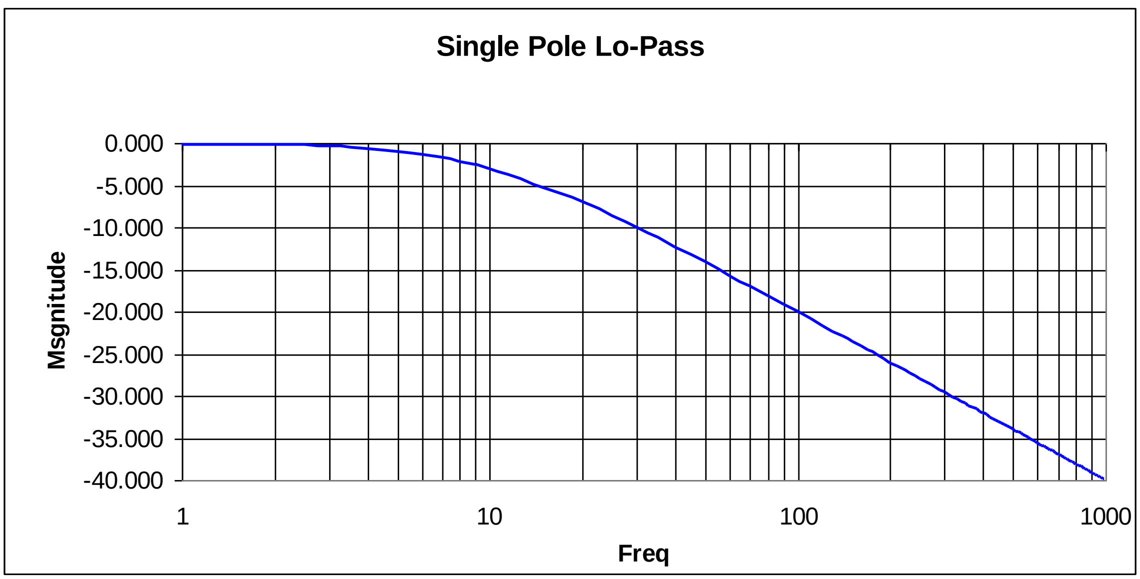
$$\frac{V_{out}}{V_{in}} = \frac{10}{s+10}$$

And by the process developed above:

$$Magnitude(\omega) = \frac{1}{\sqrt{1+\frac{\omega^2}{100}}} \rightarrow 20\log(1) - 20\log\left(\sqrt{1+\frac{\omega^2}{100}}\right)$$

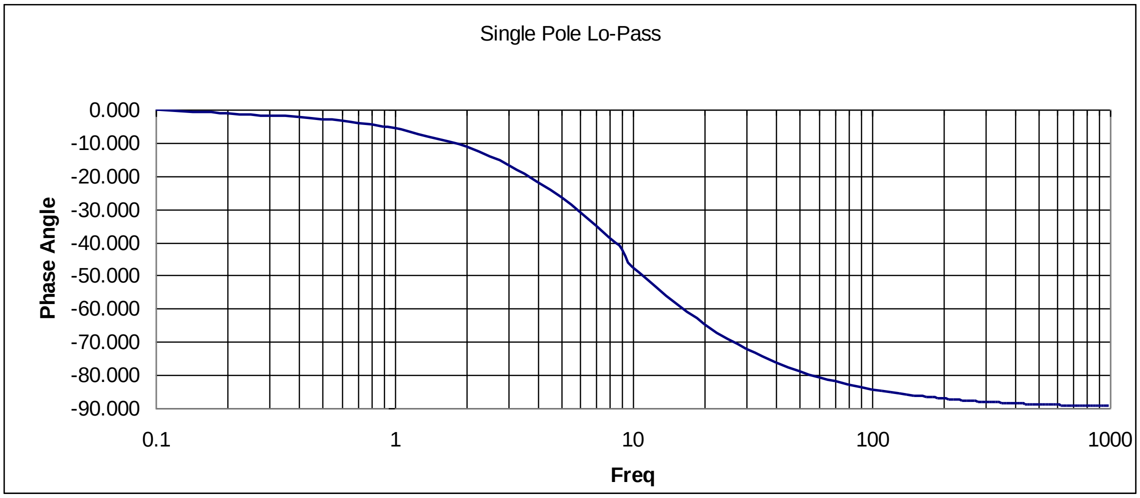
$$phase(\omega) = \arctan(0) - \arctan\left(\frac{\omega}{10}\right)$$

Briefly sketching the magnitude: when $\omega << 10$, the magnitude is $\approx 0db$ FAPP[2]; at $\omega = 10$, the magnitude is $-20\log\sqrt{2} = -3db$; when $\omega >> 10$, the magnitude for all practical purposes is $-20\log\left(\frac{\omega}{10}\right)$ (with a $-20db$.per decade slope, or "roll-off"). The "**break frequency**" is the frequency at which $\omega = \frac{1}{RC} = \omega_b$, the **-3db point**.



As can be seen from the above Bode[3] (Bo-dee) plot the output magnitude begins to roll off as ω approaches the break frequency (in this case 10 rads/s). At the break frequency, the magnitude is at -3db and rolls off at -20db per decade. Please note that this technique plots the output magnitude versus frequency, but does not address the frequency content of the output. That aspect of system response is left to Fourier analysis which is the subject of a different set of modules.

[2] For All Practical Purposes
[3] Named for H. W. Bode, 1905-1982, who developed the technique

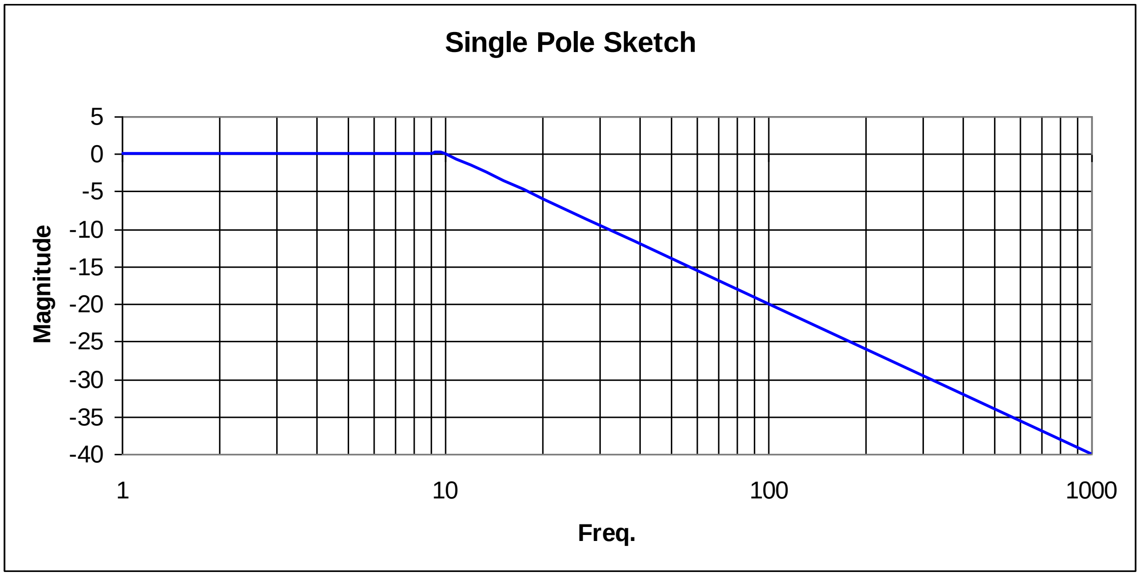


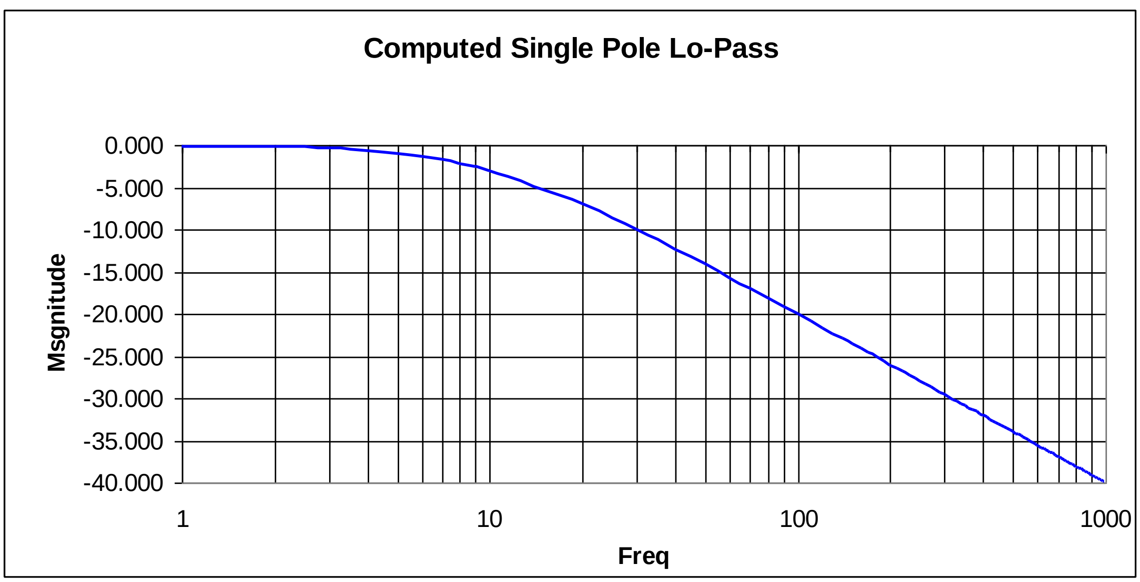
As expected for this circuit, the phase of the output with respect to the input varies from 0 to -90° as a function of $\omega(\arctan\infty\ is\ 90°)$. At very low frequencies the capacitor acts like an open and the input is the output. At high frequencies the capacitor begins to act like a short, and by the voltage division rule an ever larger percentage of the input is dropped across the resistor, while the phase of the voltage that is dropped across the capacitor approaches the -90° rail.

Both the magnitude and the phase plots are readily hand sketched for a rapid peek at the performance envelope. For the magnitude sketch use the following "rule of thumb":

a. begin at the lowest frequency of interest (often 0, or 1 when plotting logarithm format) , and find the magnitude of that frequency.
b. draw a line horizontally to the first break frequency. If the lowest freq is also a break frequency draw a line at a slope of ± 20db/decade per pole or zero (- for a pole, + for a zero).
c. from the break frequency, to the next (or to the terminal frequency if there are no further break frequencies) draw -20db/decade per pole sloped lined or a+20db/decade per zero sloped line $\left(\pm 20N\log\sqrt{1+\left(\frac{\omega}{\omega_o}\right)^2}\right)$ where N is the number of zeros or poles at that frequency.
d. repeat step c. until all break frequencies are accounted for.

Examine both the sketched and the computed plot below. The greatest error occurs at the break frequency of 10 rads/s; the computed magnitude is -3db from the magnitude sketch. That is easy to remember, your error is max at the break frequency and it is $\pm 3db$ per pole or zero at that frequency.





Rule of thumb for sketching the phase plot is:

a. Since $\tan^{-1}\frac{\omega}{\omega_o} = 1$ when $\omega = \omega_o$ assign +n45° to a zero break frequency and a -n45° to a pole break frequency. n being the number of poles or zeros at that break frequency.

b. one decade back from any break frequency assign the phase to 0 change from the previous break frequency for that pole or zero (the origin can be considered a "previous break frequency" for this purpose).

c. one decade above the break frequency assign the phase to be $+n90^o$ for those zero(s) or $-n90^o$ for those pole(s) at the break frequency.

d. connect the dots. At points of ambiguity (for example a frequency that is one decade above a pole ω_o while simultaneously being one decade back from a zero ω_o) it is **best** to compute the value. Phase plots can be very tricky to sketch, so calculation where ambiguity exists is recommended.

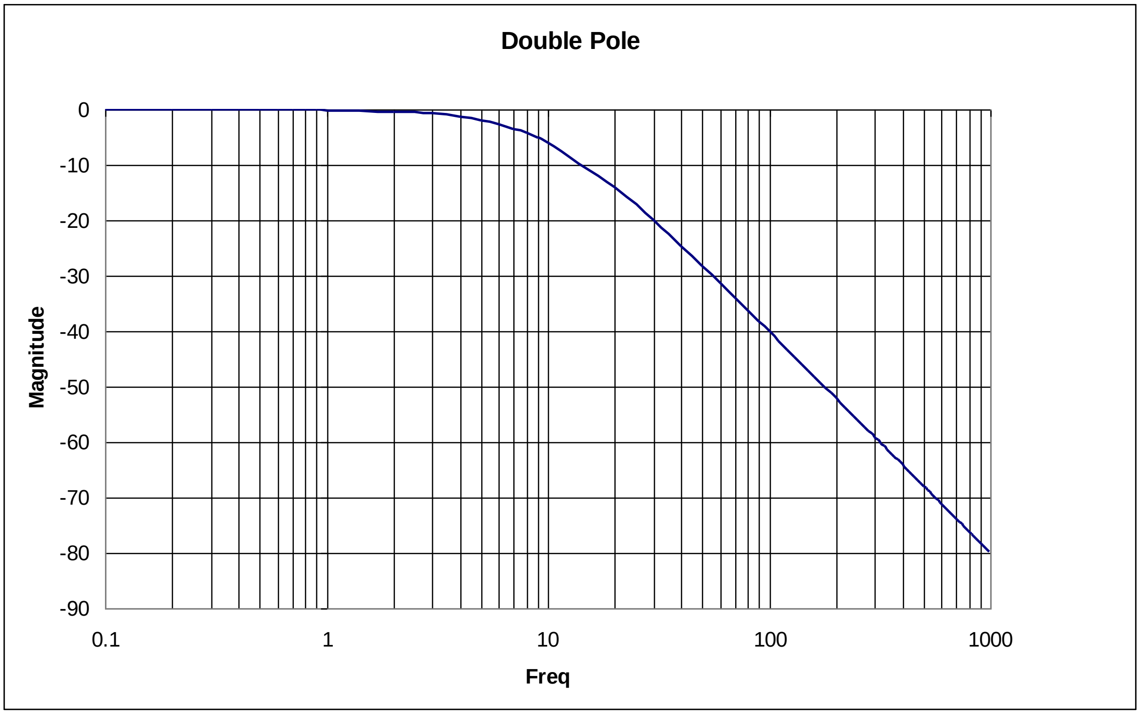
Let us consider the double pole transfer function with a break frequency of 10 rads/s (remember, for all practical purposes the poles need not be exactly superimposed, merely close enough so that treating them as exact has no appreciable effect on the outcome prediction).

$$\frac{Output}{Input} = \frac{100}{(s+10)^2}$$

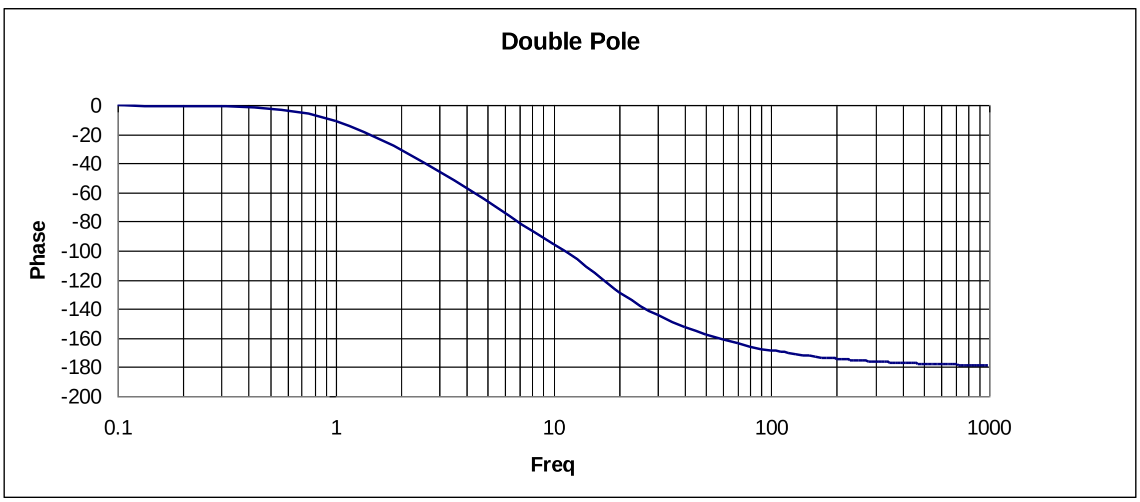
$$Magnitude(\omega) = 20\log(1) - 40\log\left(\sqrt{1+\frac{\omega^2}{10^2}}\right)$$

$$Phase(\omega) = \tan^{-1}(0) - \tan^{-1}\left(\frac{\omega}{10}\right) - \tan^{-1}\left(\frac{\omega}{10}\right)$$

Note the differences between this example and that of the single pole; the roll-off is twice as steep and the phase difference at any frequency is doubled.



Also notice that at the break frequency the magnitude is down by -6db, i.e., -3dbfor each pole.



Again the phase begins a zero, but ends at -180, or double the single pole, and is verified by the phase equation above. That will hold throughout, n poles equals output phase tending towards n*(-90). Conversely, n zeros will drive the output phase towards n*(+90).

For design and analysis purposes, when the number of zeros are ≥ the number of poles division is applied until the number of zeros is one less than the number of poles. This rule is consistent throughout all the processes of design and analysis involving transforms. When the number of zeros dominate instability is always a looming threat. However for Bode analysis this rule is set aside as division will yield as least one δ(t) and the log of the impulse is zero because its transform is 1.

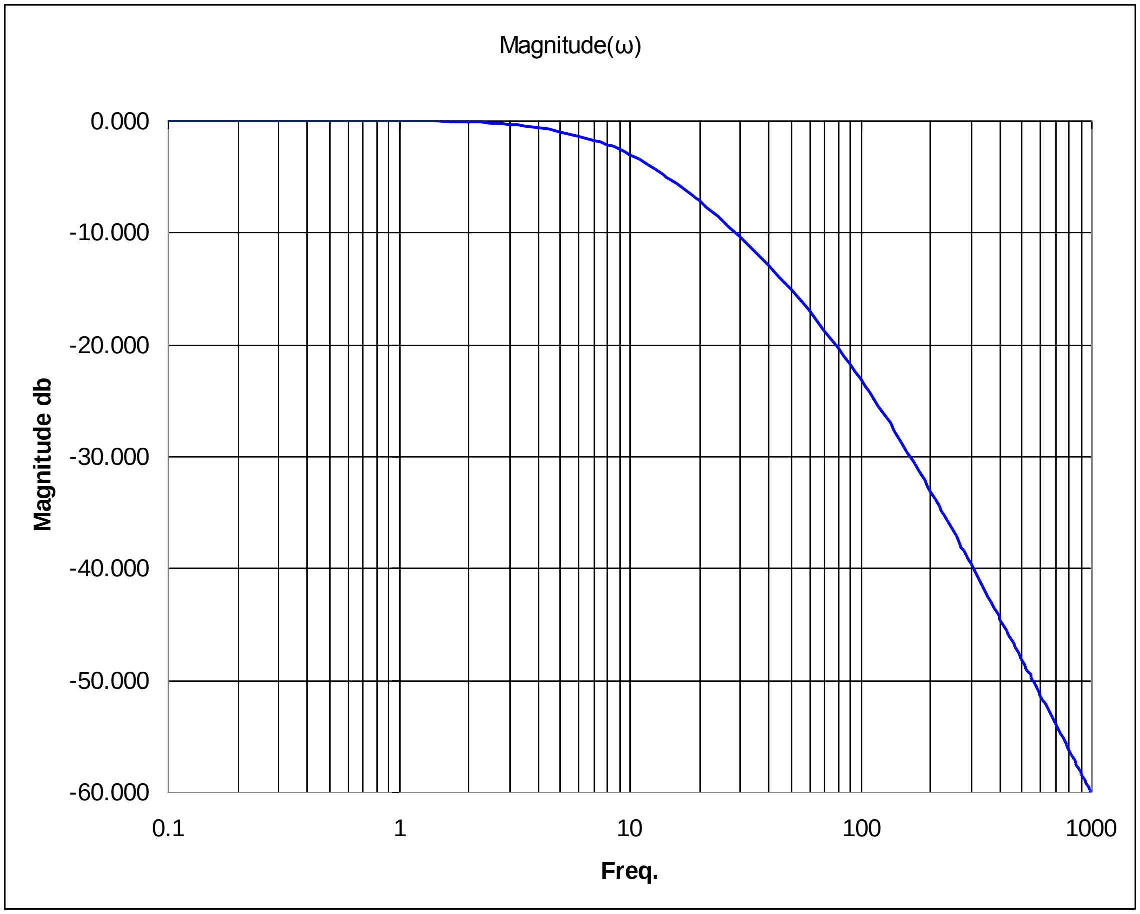
Consider a double pole filter that has two separate and distinct break frequencies, i.e., 10 rads/s and 100 rads/s. Assume the numerator is 1000 (for illustration purposes only). Then:

$$\frac{Output}{Input} = \frac{1000}{(s+10)(s+100)} = \frac{1}{\left(1+\frac{s}{10}\right)\left(1+\frac{s`}{100}\right)}$$

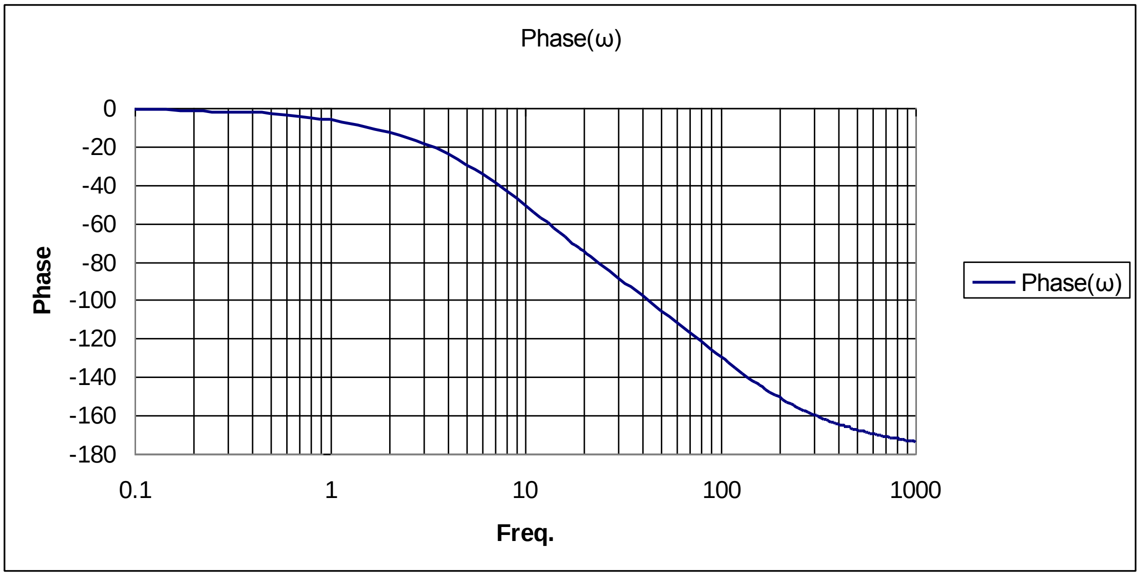
Forming the necessary equations for plotting:

$$Magnitude = 20\log(1) - 20\log\left(\sqrt{1+\frac{\omega^2}{10^2}}\right) - 20\log\left(\sqrt{1+\frac{\omega^2}{100^2}}\right)$$

$$Phase = \tan^{-1}(0) - \tan^{-1}\left(\frac{\omega}{10}\right) - \tan^{-1}\left(\frac{\omega}{100}\right)$$



As expected the roll-off between 10 and 100 rads/s s is -20db per decade, whereas the roll-off between 100 and 1000 rads/s is -40db per decade. The pattern that is emerging is that each pole contributes a -20db per decade roll-off beginning at the break frequency. If a -100db per decade roll-off is needed, then you will need a 5 pole filter. -ndb per decade requires n/20 poles.



Again, another pattern that is emerging is that each pole contributes a -90^o phase shift at the output. n poles = n*(-90^o) shifts. That fact allows you to predict the terminal phase shift as $\omega \to \infty$.

Bandwidth and Half-Power Points

Notice that the magnitude -3db at the break frequency ($\omega = \omega_b$). -3db is known as the half-power point since 10 log(.5)= -3. Filter bandwidth is usually defined as the range of frequencies between half-power points. In notch and bandpass filters there will be a pair of half-power points; one each for roll-on and roll-off. In the case of the lo-pass filter as above there is only one roll-off half-power point. More will be mentioned of this topic later in the module.

Adding Zeros to the Transfer Function

Suppose there exists a transfer function such as:

$$\frac{K(s+\alpha)}{(s+\beta)}$$

Using the procedures we have already developed the expressions necessary to plot magnitude vs frequency and phase shift vs frequency are:

$$Magnitude = 20\log(K) + 20\log\left(\sqrt{1+\frac{\omega^2}{\alpha^2}}\right) - 20\log\left(\sqrt{1+\frac{\omega^2}{\beta^2}}\right)$$

$$Phase = \tan^{-1}(0) + \tan^{-1}\left(\frac{\omega}{\alpha}\right) - \tan^{-1}\left(\frac{\omega}{\beta}\right)$$

Since the number of zeros equals the number poles division would be applied, but for purposes of Bode analysis the δ(t) is ignored as its transform is 1 and its log is zero. For example:

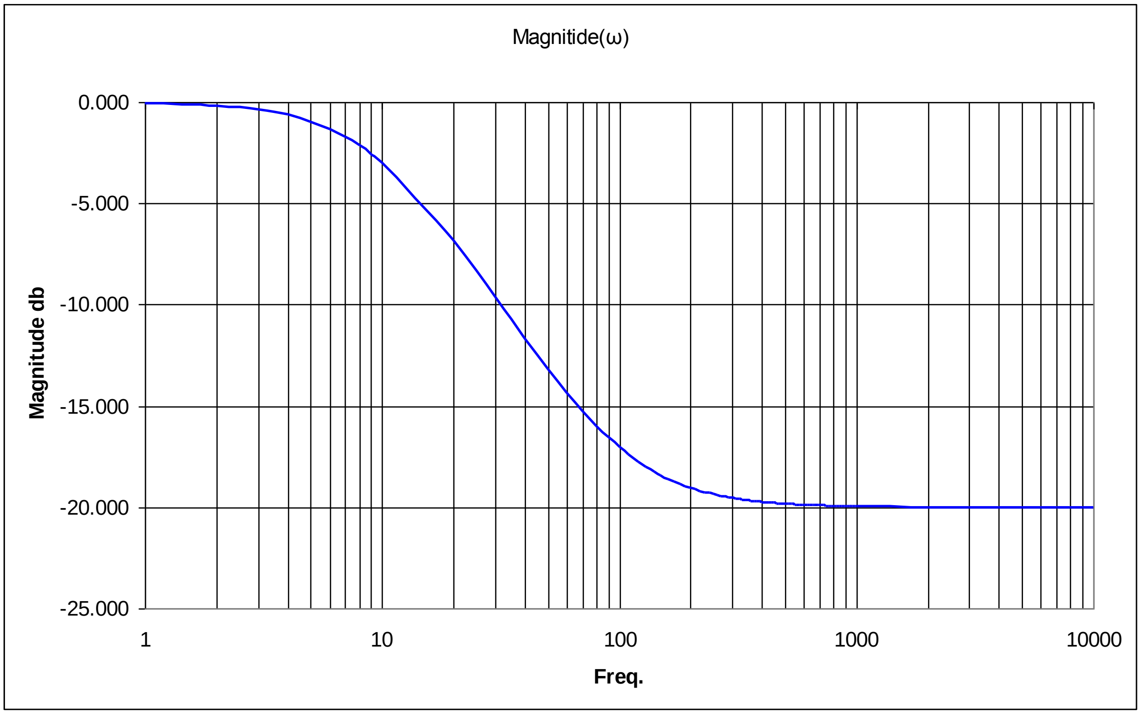
$$\frac{10(s+100)}{(s+10)} = \frac{\left(1+\frac{s}{100}\right)}{\left(1+\frac{s}{10}\right)}$$

and then:

$$Magnitude = 20\log(1) + 20\log\left(\sqrt{1+\left(\frac{\omega}{100}\right)^2}\right) - 20\log\left(\sqrt{1+\left(\frac{\omega}{10}\right)^2}\right)$$

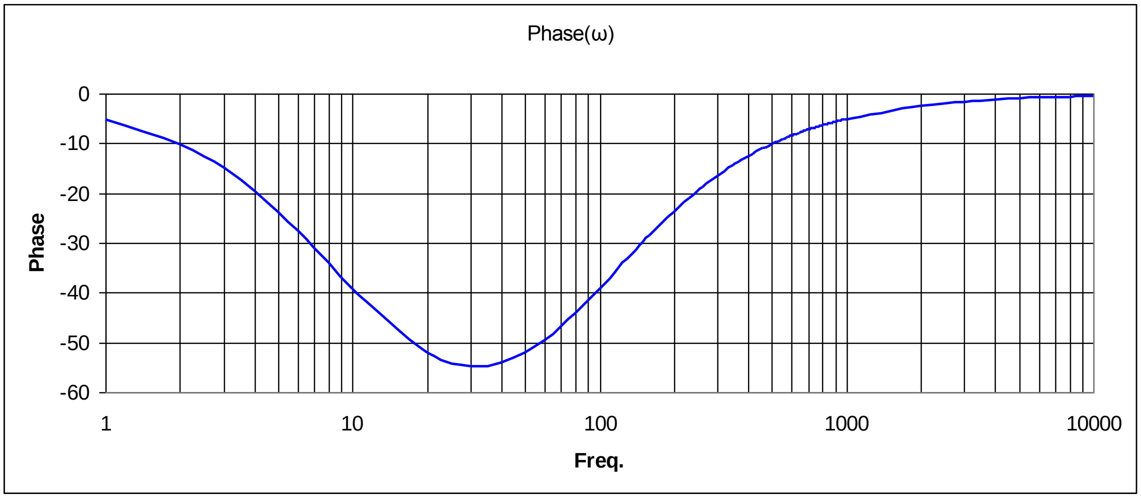
$$Phase = \tan^{-1}(0) + \tan^{-1}\left(\frac{\omega}{100}\right) - \tan^{-1}\left(\frac{\omega}{10}\right)$$

Plotting these:



From the expression that the magnitude versus frequency plot is constructed, we should expect a -20db roll-off beginning at 10 rads/s and a +20db roll-on at 100 rads/s.

From the above graph those two effects can be seen. Since the +20db term cancels the -20db term beginning at 100 rad break frequency the magnitude of the output with respect to the input remains constant out beyond about 500 rads/s. The net effect of the zero is to cancel the effect of the pole at the zero break frequency. This effect is very useful in designing and constructing bandpass and notch filters.



It was simple to predict that the terminal phase change would sum to zero as $\omega \to \infty$, since -90°+90°=0. Since the two break frequencies are 10 rads/s and 100 rads/s we should expect maximum phase changes $\left(\frac{d\phi}{d\omega}\right)$ to occur within that range of frequencies; from the plot it is seen that there is indeed a minimum accompanied by a sign change.

Since voltage division must be true regardless of the technique in use, consider the case of a simple RC series circuit (again let RC=.1) with output taken across the capacitor. Recalling that $X_C = \frac{-j}{\omega C}$, the transfer function as a function of ω from inspecting the circuit schematic is:

$$\frac{V_{out}}{V_{in}} = \frac{-j}{\omega C\left(R - \frac{j}{\omega C}\right)} = \frac{-j}{(\omega RC - j)} = \frac{-j}{(.1\omega - j)} \quad \leftarrow \text{transfer function}$$

Then as a function of ω, the phase starts out as zero when $\omega = 0$, and:

$$phase(\omega) = \arctan(0) - \arctan\left(\frac{\omega}{10}\right)$$

as verified by inspection of the phase equation. The phase ends at -90^o as $\omega \to \infty$, again as verified by inspection of the circuit and the equation. The magnitude as a function of ω is again found by the use of the Pythagorean theorem and using absolute values for j. So:

$$\text{output equals } 20\log(10) - 20\log\left(\sqrt{1+\frac{\omega}{10}}\right)$$

Finding peak/valley maximums and minimums between break frequencies is done by computation using the discussed techniques for creating both the magnitude and phase equations. Let your calculator do the work, its only essential that you understand the principles so that you can design the circuit to accomplish the aims. Exercising the equations and varying break frequencies, and/or time constants allows you to become familiar with the effect. For example, the further apart break frequencies get, the higher the peak amplitude (try it for yourself). The closer break frequencies are to each other "sharpens" the peak; a sharp peak infers that the circuit has greater selectivity than a blunt peak. Selectivity being a measure of acquiring the frequency of choice while attenuating other frequencies. It will be informative to you exercise the equations, and discover how things change as a function of break frequency and time constant.

The Complex Quadratic Pole

Consider a transfer function such as:

$$\frac{Output}{Input} = \frac{N(s)}{s^2 + \frac{b}{a}s + \frac{c}{a}}$$

where $\frac{b}{a}$ & $\frac{c}{a}$ are such that the roots are complex. Let us change the notation to something more convenient to our purposes; let $\frac{b}{a} = 2\zeta\omega_0$ and $\frac{c}{a} = \omega_o^2$, then:

$$\frac{Output}{Input} = \frac{N(s)}{s^2 + 2\zeta\omega_0 s + \omega_o^2}$$

In order to remain complex restrict $0 \le \zeta < 1$.

Substituting $j\omega$ for s:

$$\frac{Output}{Input} = \frac{N(s)}{\left(\omega_o^2 - \omega^2\right) + j2\zeta\omega_0\omega}$$

$$\frac{Output}{Input} = \frac{N'(s)}{\left(1 - \frac{\omega^2}{\omega_o^2}\right) + j2\zeta\frac{\omega}{\omega_o}}$$

As an example, set $N'(s) = 1$ and $\omega_o = 1$. Since the salient effect occurs as a function of ζ (zeta), plots will be made for a few different values of ζ.

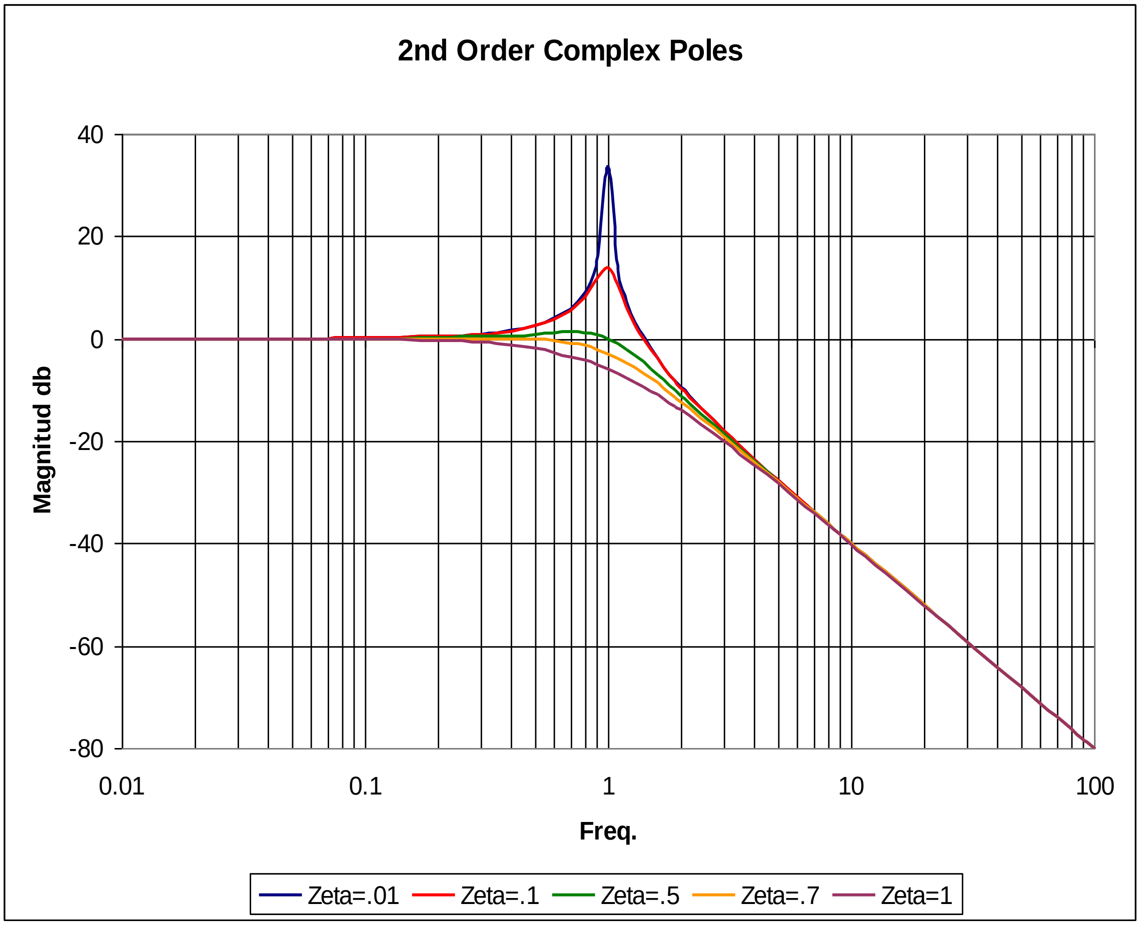
$$Mag. = 20\log(1) - 20\log\left(\sqrt{\left(1 - \frac{\omega^2}{1}\right)^2 + \left(2\zeta\frac{\omega}{1}\right)^2}\right)$$

$$Phase = 0 - \tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_o}}{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)}\right)$$

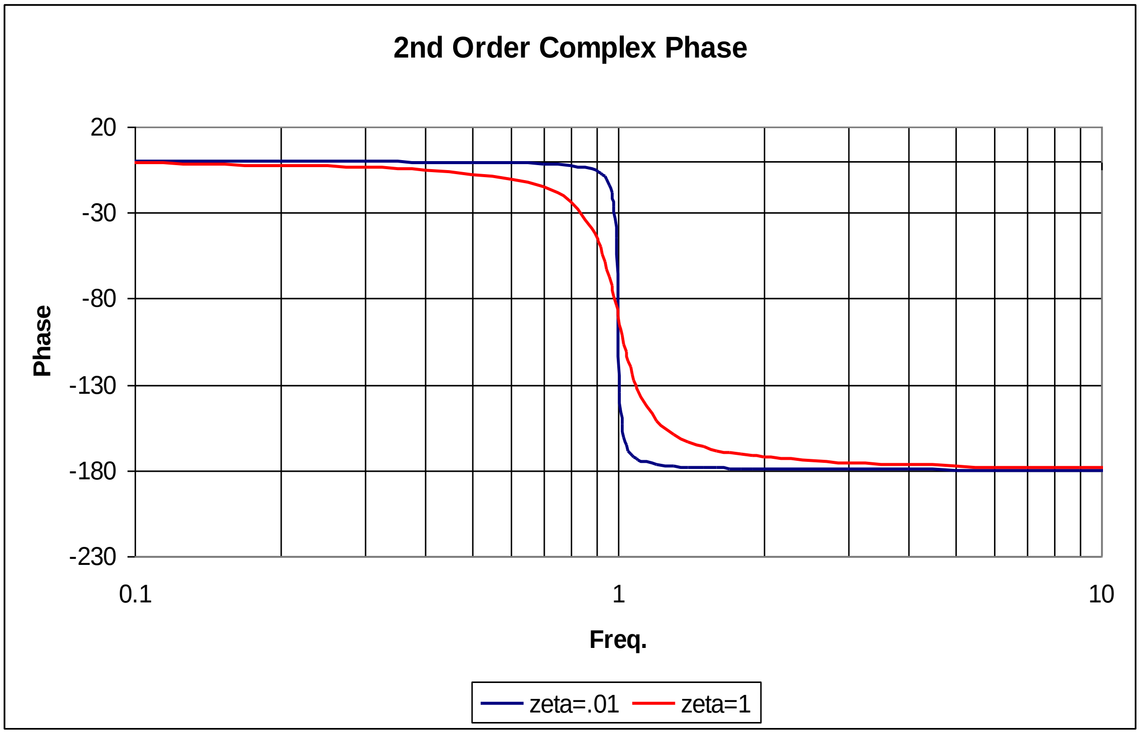
Use the above when the denominator is positive. Remember at ω_o the phase is -90^o, avoiding division by zero.

$$Phase = -180 - \tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_o}}{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)}\right)$$

Use the above when the denominator is negative.

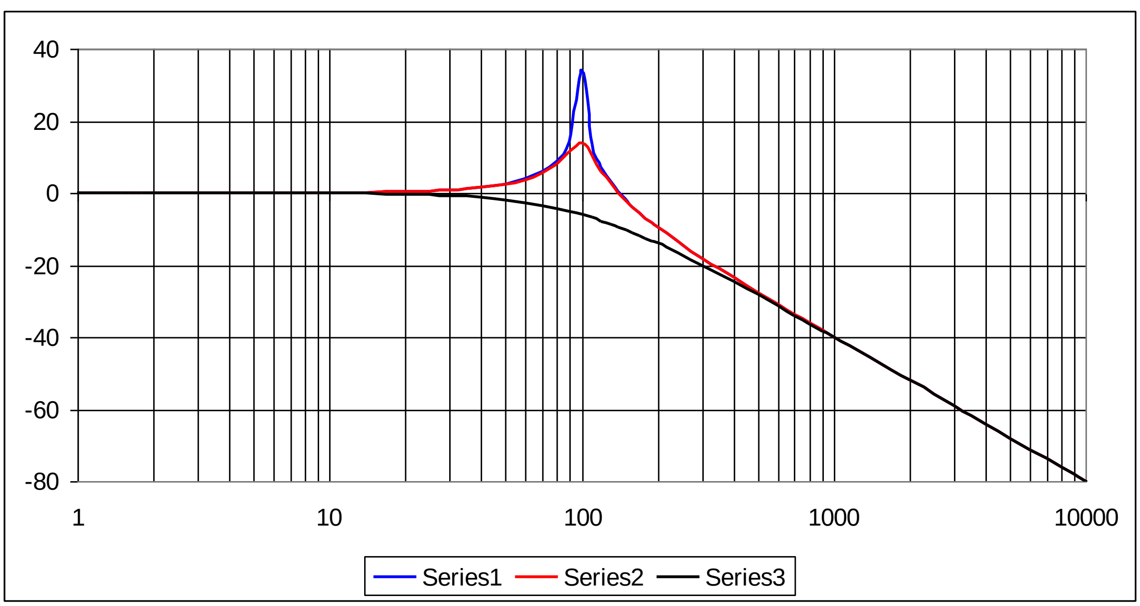


We still get the -40 db/decade roll-off, but the response of the system about the break frequency varies significantly as a function of zeta. When zeta=.01, there exists an extremely narrow selectivity bandwidth about the break frequency. When zeta=.01 the time constant, in this case, equals 100s and the system impulse response dies out at 600s. When $\zeta = 1$, the system is just a 2nd order repeated real pole filter at a break frequency of 1.



When the resonant frequency is changed, all that changes in the pattern is the frequency that the peaks occur at in the magnitude plot, and the frequency that holds the -90^o spot in the phase plot.

For example:



Since we have used the notation $s^2 + 2\zeta\omega_o s + \omega_o^2$, and since we restrict this notation to those cases wherein the roots are complex, it is legitimate to ask about the relationship between the resonant frequency (ω_o) and the damped frequency (ω_d) as a function of ζ (we will label ζ the damping coefficient from here on out to the end of the series of modules). Clearly when $\zeta = 0$ the equation becomes $s^2 + \omega_o^2$ and the roots are $\pm j\omega_o$ and that is the resonant frequency and the frequency of oscillation. However when $0 < \zeta < 1$, then the roots of the equation are $-\zeta\omega_o \pm j\sqrt{\omega_o^2 - (\zeta\omega_o)^2}$, where the frequency of oscillation $\omega_d = \sqrt{\omega_o^2 - (\zeta\omega_o)^2}$ or:

$$\omega_d = \omega_0\sqrt{1-\zeta^2}$$

Check it out; let a denominator be $s^2 + 4s + 13$. The roots are $s = -2 \pm j3$, $\omega_o = \sqrt{13}$, $\omega_d = 3$ and $4 = 2\zeta\omega_o$; therefore $\zeta = \dfrac{2}{\sqrt{13}}$. So:

$$3 = \sqrt{13}\sqrt{1-\left(\frac{2}{\sqrt{13}}\right)^2} = \sqrt{13-4} = \pm 3 \quad \text{Q.E.D.}$$

So ends Module 4.